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Bracket and Exponential for
a New Type of Vector Field

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In [2] Robert Hermann introduced the concept of
tangent vector fields on the space of functions from
one manifold to another. He applied these to give a
new proof of the Cartan-Kahler theorem. An example
of such vector fields are maps from the jet space to
the tangent bundle of the target space which commute
with projections. It is this class of vector fields
which we study here.

Using prolongations a Lie bracket operation is
defined and justified on the grounds that it agrees
with the primitive definition when the latter has
meaning here. By similar methods an exponential ex-
pansion is deduced. An example is given which shows
that the 1-parameter transformation groups on the
function space cannot be considered a parameter space
for a pseudo group in Kuranishi's sense [3], for it
need not involve infinite analytic mappings.

1. Introduction

Every mapping and manifold will be smooth of
class C^∞ unless otherwise noted. If N and M

are two manifolds, $J^k = J^k(N,M)$ is the manifold of
k-jets $j_x^k(f)$ of order k of maps $f:N \to M$ (see
[1]). $\underline{\alpha}$ and $\underline{\beta}$ denote the customary source and
target projections. $T(M)$ is the tangent bundle of
M, M_y the tangent space at $y \in M$. $\underline{\pi}:T(M) \to M$
is the bundle projection. $C^\infty(N,M)$ is the set of
all C^∞ maps on N into M.

Definition 1. A k-<u>vector</u> <u>field</u> <u>on</u> $C^\infty(N,M)$
is a map $\underline{\Theta}:J^k \to T(M)$ such that $\underline{\pi} \circ \underline{\Theta} = \underline{\beta}$.

Hermann studied k-vector fields as a special class
of " formal tangent vector " fields [2, p. 8]. If
$f:N \to M$, a " vector " along f is a map $\underline{\psi}:N \to T(M)$
with $\underline{\psi}(x) \in M_{f(x)}$ for all $x \in N$. This is what one

would get as the derivative of a 1-parameter family
$f_t \in C^\infty(N,M)$ where $f_0 = f$. Each k-vector field
$\underline{\Theta}$ defines a vector along f by $\underline{\psi}(x) = \Theta(j_x^k(f))$.

Let $I = (-\underline{\epsilon}, \underline{\epsilon})$. An <u>integral</u> <u>curve</u> of $\underline{\Theta}$
starting at $f_0 \in C^\infty(N,M)$ is a 1-parameter family
of $f:N \times I \to M$ with $f(x,0) = f_0(x)$ and

$$\frac{\partial f}{\partial t}(x,t) = \underline{\Theta}(j_x^k(f)).$$

In coordinates this is seen to be a Cauchy-Kowalewski
system of order k. By uniqueness of C^ω (i.e., analytic)
solutions we see that in the C^ω case if $g_0(x) = f(x,t)$,

and $g(x, \tau)$ is an integral curve starting at g_o, then $g(x, \tau) = f(x, t + \tau)$. Thus, these integral curves, when they exist and are unique, behave as the orbits of a local 1-parameter group.

2. An Example

Let $N = M = E^1$, Euclidean 1-space. Let (x), (y), and (x,y,p) be coordinates on N, M, and $J^1(N,M)$, respectively. Consider the 1-vector field $\underline{\varrho}(x,y,p) = p(\partial / \partial y)_y$. Given $f_o(x):N \to M$, $f(x,t)$ must satisfy

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} , \quad f(x,0) = f_o(x).$$

We think of $f(x,t)$ as the image of f_o under a transformation F_t on functions: $f(x,t) = F_t(f_o)$.

To compare the action of F_t with M. Kuranishi's concept of infinite analytic mappings [3], consider all convergent power series at the origin: $f_o = \sum a_n x^n$. Then

$$F_t(f_o) = \sum_{n=0} (\sum_{m=0} \frac{t^m}{m!} a_{n+m} \frac{(n+m)!}{n!})x^n$$

is not an infinite analytic mapping in Kuranishi's sense because the coefficient of x^n in $F_t(f_o)$ is an infinite series in the coefficients of f_o rather than a polynomial.

3. Lie Bracket

If $\underline{Q}$ is a k-vector field let $h_t: J^k \to M$ for $-\underline{\xi} < t < \underline{\xi}$ satisfy $h_o = \underline{\beta}$ and $(\partial h_t / \partial t)_{t=0} = \underline{Q}$. If $f \in C^\infty(N,M)$, the kth prolongation of f, $p^k(f) \in C^\infty(N,J^k)$ is defined by $p^k(f)(x) = j_x^k(f)$. Similarly, the rth prolongation of h_t, $p^r(h_t) \in C^\infty(J^{k+r},J^r)$, is defined by

$$p^r(h_t)(j_x^{r+k}(f)) = j_x^r(h_t \circ p^k(f)).$$

Definition 2. $P^r(\underline{Q})$, the rth <u>prolongation</u> of $\underline{Q}$, is defined to be

$$P^r(\underline{Q})(j_x^{r+k}(f)) = \frac{\partial}{\partial t} [p^r(h_t)(j_x^{r+k}(f))]_{t=0} .$$

Then $P^r(\underline{Q}): J^{r+k} \to T(J^r)$, and if π denotes the projection of $T(J^r)$ onto J^r and $\mathcal{L}_r^{r+k}$ carries $j_x^{k+r}(f)$ in J^{k+r} to $j_x^r(f)$ in J^r, then $\underline{\pi} \bullet P^r(\underline{Q}) = \mathcal{L}_r^{r-k}$.

Kuranishi's notion of formal partial derivative is very useful in describing $P^r(\underline{Q})$ in coordinates. If $(x^1,\dots,x^n)$ and $(y^1,\dots,y^m)$ are local coordinates on $U \subset N$ and $V \subset M$, respectively, let

$(x^i, y^\lambda, p^\lambda{}_{j_1}, \ldots, p^\lambda{}_{j_1 \ldots j_k})$ be local coordinates on

$\underline{\alpha}^{-1}(U) \cap \underline{\beta}^{-1}(V)$, where $\lambda = 1, \ldots, m$; $i_h = 1, \ldots, n$.

If $u: J^k \to R$ (real numbers), define $\delta_j^\# u: J^{k+1} \to R$ to be

$$\delta_j^\# u(J_x^{k+1}(f)) = \frac{\partial u}{\partial x^j} + \frac{\partial u}{\partial y^\lambda} p^\lambda{}_j + \ldots + \frac{\partial u}{\partial p^\lambda{}_{j_1 \ldots j_k}} p^\lambda{}_{j_1 \ldots j_k j} \ .$$

This operator is linear and has the important property
that if $f: U \to V$, then $\partial(u \circ p^k(f))/\partial x^j = \delta_j^\# u(p^{k+1}(f))$.
Using these facts and Definition 2, it is possible to
prove the following

Lemma 1. **If on** $\underline{\alpha}^{-1}(U) \cap \underline{\beta}^{-1}(V)$, $\underline{\varrho} = \underline{\varrho}(\partial/\partial y^\lambda)$,

$$p^r \underline{\varrho} = \underline{\varrho}^\lambda \frac{\partial}{\partial y^\lambda} + \delta_{j_1}^\# \underline{\varrho}^\lambda \frac{\partial}{\partial p^\lambda{}_{j_1}} + \ldots + \delta_{j_1}^\# (\ldots \delta_{j_r}^\# (\underline{\varrho}^\lambda) \frac{\partial}{\partial p^\lambda{}_{j_1 \ldots j_r}} \ .$$

Moreover, p^r **is linear.**

Definition 3. Let $\underline{\varrho}$ and $\underline{\psi}$ be r- and s-vector
fields, respectively. Then the **Lie bracket** of $\underline{\varrho}$ and $\underline{\psi}$
is

$$[\underline{\varrho}, \underline{\psi}] = p^s \underline{\varrho} \circ \underline{\psi} - p^r \underline{\psi} \circ \underline{\varrho} \ .$$

By the composition notation we mean to interpret $\underline{\psi}$ as an
operator carrying $C^\infty(M, R)$ into $C^\infty(J^s, R)$. Similarly,

$P^s\underline{\varrho}$ carries $C^\infty(J^s,R)$ into $C^\infty(J^{r+s},R)$.

Alternatively, if $h_t:J^r \to M$ and $g_\tau:J^s \to M$ satisfy $h_0 = \underline{a}$, $g_0 = \underline{a}$, $-\underline{\epsilon} < t < \underline{\epsilon}$, $-\underline{\epsilon} < \tau < \underline{\epsilon}$, and $(\partial h_t/\partial t)_{t=0} = \underline{\varrho}$, $(\partial g_\tau/\partial \tau)_{\tau=0} = \underline{\psi}$, then

$$(P^s\underline{\varrho} \circ \underline{\psi})(j_x^{r+s}(f)) = \frac{\partial^2}{\partial t \partial \tau}[g_\tau \circ P^s h_t(j_x^{r+s}(f))]_{\tau = t = 0}.$$

This representation is convenient when proving

Lemma 2. $[\underline{\varrho},\underline{\psi}]$ is an $r + s$-__vector field__.

Lemma 3. $F^q(P^x\underline{\varrho} \circ \underline{\psi}) = P^{x+q}\underline{\varrho} \circ F^q\underline{\psi}$.

Proof. This follows from $P^q(g_\tau \circ P^s h_t) = P^q g_\tau \circ P^{s+q} h_t$, which is a consequence of the definitions.

Lemma 4. (Lie Identity)

$$[[\underline{\varrho},\underline{\psi}]\underline{\varphi}] + [[\underline{\psi},\underline{\varphi}]\underline{\varrho}] + [[\underline{\varphi},\underline{\varrho}]\underline{\psi}] = 0.$$

Proof. Suppose $\underline{\varrho}$, $\underline{\psi}$, $\underline{\varphi}$ are r-, s-, and q-vector fields, respectively. The left side of the above equation is

$$P^q(P^x\underline{\varrho} \circ \underline{\psi} - P^r\underline{\psi} \circ \underline{\varrho})\circ\underline{\varphi} - P^{s+r}\underline{\varphi} \circ (I^s\underline{\varrho} \circ \underline{\psi} - P^r\underline{\psi} \circ \underline{\varrho})$$

$$+ P^r(P^q\underline{\psi} \circ \underline{\varphi} - P^s\underline{\varphi} \circ \underline{\psi}) \circ \underline{\varrho} - P^{q+s}\underline{\varrho} \circ (F^q\underline{\psi} \circ \underline{\varphi} - P^s\underline{\varphi} \circ \underline{\psi})$$

$$+ P^s(P^r\underline{\varphi} \circ \underline{\varrho} - P^q\underline{\varrho} \circ \underline{\varphi}) \circ \underline{\psi} - P^{r+q}\underline{\psi} \circ (F^r\underline{\varphi} \circ \underline{\varrho} - P^q\underline{\varrho} \circ \underline{\varphi}).$$

This equals zero by the __Lemma__ 3 above.

It follows from Definition 3 of the Lie bracket that $[\underline{\varrho}, \underline{\varrho}] < 0$. Hence we have proved

Theorem 1. **If** V^k **is the linear space of all**
k-**vector fields**, k = 0,1,..., **then the direct sum**

$$V = \sum_{k=0}^{\infty} \bullet V^k$$

is a graded Lie algebra under the Lie bracket.

A definition of Lie bracket in the older litera-
ture used local 1-parameter transformation groups.
If $\underline{\Theta}$ and $\underline{\Psi}$ are vector fields on a manifold
generating local transformation groups $\underline{\Theta}_t$ and $\underline{\Psi}_r$
respectively, then $[\underline{\Theta}, \underline{\Psi}]$ is the vector field ob-
tained by transforming $\underline{\Theta}_t$ by $\underline{\Psi}_r$. That is,

$$[\underline{\Theta}, \underline{\Psi}] = \frac{\partial^2}{\partial t\, \partial \tau} [\underline{\Psi}_r \circ \underline{\Theta}_t \circ \underline{\Psi}_{-\tau}]_{t=\tau=0} \quad .$$

We do not have 1-parameter local groups on $C^\infty(N,M)$
in general. However, one may observe how individual
functions behave when they belong to integral curves
of $\underline{\Theta}$ and $\underline{\Psi}$.

Theorem 2. **Let** $\underline{\Theta}$ **and** $\underline{\Psi}$ **be** r- **and** s-**vector**
fields, respectively. Let $I = (-\underline{\varepsilon}, \underline{\varepsilon})$ **. Let**
$f(x):N \to M$. **Suppose:**

a. $\overline{f}(x,t):N \times I \to M$ **and satisfies**

$$\frac{\partial \overline{f}}{\partial t}(x,t) := \underline{\Psi}(j_x^r(\overline{f})) \, , \quad \overline{f}(x,0) = f(x);$$

b. $f*(x,t,\underline{\tau}):N\times I\times I \rightarrow M$ **and** **satisfies**

$$\frac{\partial f*}{\partial t}(x,t,\underline{\tau}) = \underline{\varphi}(j_x^s(f*)), \quad f*(x,t,0) = \overline{f}(x,t);$$

c. $\overline{f}*(x,t,\underline{\tau},\underline{t}):N\times I\times I\times I \rightarrow M$ **and** **satisfies**

$$\frac{\partial f*}{\partial t}(x,t,\underline{\tau},\underline{t}) = \underline{\psi}(j_x^r(\overline{f}*)), \quad \overline{f}*(x,t,\underline{\tau},0) = $$
$$f*(x,t,\underline{\tau}) \quad .$$

Then

$$\frac{\partial^2}{\partial t\partial\underline{\tau}}\overline{f}*(x,-t,\underline{\tau},t)\Big|_{t=\underline{\tau}=0} = [\underline{\varphi},\underline{\psi}](j_x^{s+r}(f)) \quad .$$

Proof.

$$\frac{\partial^2}{\partial t\partial\underline{\tau}}\overline{f}*(x,-t,\underline{\tau},t)\Big|_{t=\underline{\tau}=0} = -\frac{\partial^2}{\partial t\partial\underline{\tau}}\overline{f}*(x,t,\underline{\tau},\underline{t})\Big|_{t=\underline{\tau}=\underline{t}=0}$$

$$+\frac{\partial^2}{\partial\underline{\tau}\partial\underline{t}}\overline{f}*(x,t,\underline{\tau},\underline{t})\Big|_{t=\underline{\tau}=\underline{t}=0}$$

$$= -\frac{\partial^2}{\partial t\partial\underline{\tau}}f*(x,t,\underline{\tau})\Big|_{t=\underline{\tau}=0} + \frac{\partial}{\partial\tau}\underline{\psi}[j_x^r(f*)]_{t=\underline{\tau}=0}$$

$$-\frac{\partial}{\partial t}\underline{\varphi}(j_x^s(\overline{f}))\Big|_{t=0} + \frac{\partial}{\partial\tau}\underline{\psi}[j_x^r(f*(x,0,\underline{\tau}))]_{\underline{\tau}=0} \quad .$$

We can calculate these using local coordinates (see above).

$$\frac{\partial}{\partial t} \, \varrho^{\lambda}(x^1, \, \bar{f}^{\mu}, \, \frac{\bar{f}^{\mu}}{\partial x^j}, \ldots, \, \frac{\partial^s \bar{f}^{\mu}}{\partial x^{j_1} \ldots \partial x^{j_s}} \,)$$

$$= \frac{\partial \varrho^{\lambda}}{\partial y^{\mu}} \frac{\partial \bar{f}^{\mu}}{\partial t} + \frac{\partial \varrho^{\lambda}}{\partial p^{\mu}_j} \frac{\partial \bar{f}^{\mu}}{\partial t \partial x^j} + \ldots + \frac{\partial \varrho^{\lambda}}{\partial p^{\mu}_{j_1 \ldots j_s}} \frac{\partial^{s+1} \bar{f}^{\mu}}{\partial t \partial x^{j_1} \ldots \partial x^{j_s}}$$

$$= \frac{\partial \varrho^{\lambda}}{\partial y^{\mu}} \Psi^{\mu} + \frac{\partial \varrho^{\lambda}}{\partial p^{\mu}_j} \partial^{\#}_j \Psi^{\mu} + \ldots + \frac{\partial \varrho^{\lambda}}{\partial p^{\mu}_{j_1 \ldots j_s}} \partial^{\#}_{j_1} (\ldots \quad \partial^{\#}_{j_s} (\_))$$

$$= F \Psi \circ \varrho \ .$$

In the same way the last term in the first equation becomes $F^s \varrho \circ \Psi$. Q.E.D.

3. The Exponential Map

Lemma 5. _Let_ ϱ _and_ ψ _be_ r- _and_ s-_vectors,_
respectively. _Suppose_ $f(x,t):N\times I \to M$ _satisfies_
$\varrho(j_x^r(f)) = (\partial f/\partial t)(x)$. _Then_

$$\frac{\partial}{\partial t}\psi(j_x^s(f)) = P^s\varrho \circ \psi(j_x^{r+s}(f)) .$$

Proof. In the previous local coordinates, we
are given that $\varrho^\lambda(j^r(f)) = (\partial f^\lambda/\partial t)(x,t)$. Hence

$$\frac{\partial}{\partial t}\psi^\lambda(j_x^s(f)) = \frac{\partial \psi^\lambda}{\partial y^\mu}\frac{\partial f^\mu}{\partial t} + \frac{\partial \psi^\lambda}{\partial p^\mu_j}\frac{\partial^2 f^\mu}{\partial t\,\partial x^j} + \ldots$$

$$+ \frac{\partial \psi^\lambda}{\partial p^\mu_{j_1\ldots j_s}}\frac{\partial^{s+1} f^\mu}{\partial t\,\partial x^{j_1}\ldots \partial x^{j_s}}$$

$$= \frac{\partial \psi^\lambda}{\partial y^\mu}\varrho^\mu + \frac{\partial \psi^\lambda}{\partial p^\mu_j}\partial_j\varrho^\mu + \ldots + \frac{\partial \psi^\lambda}{\partial p^\mu_{j_1\ldots j_s}}\partial_{j_1}^{\#}(\ldots \partial_{j_s}^{\#}(\varrho^\mu)),$$

evaluated at $j_x^{s+r}(f)$. The result follows from Lemma 1.

Theorem 3. **Let $\underline{Q}$ be an r-vector field, where** $N = E^n$, $M = E^m$. **Let** $f(x,t):E^n \times I \to E^m$ **satisfy** $(\partial f/\partial t)(x,t) = \underline{Q}(j_x^r(f))$, **where**

$$f(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_n(x)$$

converges on E^n I. **Then**

$$f_n(x) = P^{(n-1)r}\underline{Q} \circ P^{(n-2)r}\underline{Q} \circ \ldots \circ P^r\underline{Q} \circ \underline{Q}(j_x^{rn} f(x,0)).$$

Proof. By repeated application of Lemma 5 and from $\partial f/\partial t = \underline{Q}(j_x^r(f))$ it follows that

$$\frac{\partial^2 f}{\partial t^2}(x,t) = P^r\underline{Q} \circ \underline{Q}(j_x^{2r}(f)),$$

$$\frac{\partial^3 f}{\partial t^3}(x,t) = P^{2r}\underline{Q} \circ P^r\underline{Q} \circ \underline{Q}(j_x^{3r}(f)), \ldots, Q.E.D.$$

Theorem 3 shows that in thinking of $\underline{Q}$ as an infinitesimal transformation on $C^\infty(N,M)$, one should consider

$$\exp(\underline{Q})(f_0) = \sum_{n=0} \frac{t^n}{n!} P^{(n-1)r}\underline{Q} \circ \ldots \circ P^r\underline{Q} \circ \underline{Q}(j_x^{rn}(f_0))$$

as a generalized exponential formula.

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References

1. C. Ehresmann, Introduction à la théorie des structures
    infinitesimales et des pseudogroupes de Lie. Colloque
    de Géométrie Différentielle de Strasbourg, C.N.R.S.
    1953, pp. 97-117.

2. R. Hermann, Formal tangency of vector fields in func-
    tion spaces, Mimeographed Notes, U. of Calif. at
    Berkeley, 1961.

3. M. Kuranishi, On the local theory of continuous
    infinite pseudo groups, I, Nagoya Math. J. 15 (1959)
    pp. 225-260.